On some new type fractional inequalities

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Abstract

In this paper, we study some new type fractional inequalities using Caputo and k-Caputo derivatives. We also investigate some useful results for this derivative. We extend some recent and classical inequalities to this interesting calculus including Hermite-Hadamard inequality.

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1 Introduction

In recent years, fractional calculus is becoming popular in mathematics and it has lot of applications for control theory, transform theory, statical problems computation analysis and engineering. Together with these developments some new definitions of fractional derivative have been introduced to provide the best method for fractional calculus. Scientists in their diverse fields develop various aspects of their studies with fractional analysis. Convex functions play an important role in the advancement of many inequalities. Many known and useful inequalities are the consequences of convex functions. The most popular one is the Hermite-Hadamard inequality which interpret convex functions. A number of generalizations and extensions have been provided for this inequality using fractional calculus. For detailed information, we refer the readers to [1]-[11]

In this paper, we study some new forms of Hermite-Hadamard fractional integrals via convex functions using Caputo fractional derivative. We start with the following necessary definition.

Definition 1.1. A function $f : [a, b] \to \mathbb{R}$ is said to be convex if the inequality

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$

holds for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. If the inequality holds in the reverse direction, the function f is called concave.

Definition 1.2. $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $u, v \in I$ with $u < v$, the inequality

$$f\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u} \int_u^v f(x)\, dx \leq \frac{f(u)+f(v)}{2}$$

is known as the Hermite-Hadamard inequality.

Definition 1.3. Suppose that $\alpha > 0,\ t > a,\ \alpha, a, t \in \mathbb{R}$. The fractional operator

$$^{C}D^{\alpha} f(t) := \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int\limits_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, & n-1 < \alpha < n \in \mathbb{N} \\ \frac{d^n}{dt^n} f(t), & \alpha = n \in \mathbb{N} \end{cases}$$

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is called the Caputo fractional derivative or Caputo fractional differential operator of order α. This operator is introduced by the Italian mathematician Caputo in 1967.

Definition 1.4. Let $\alpha > 0$ and $\alpha \notin \{1,2,3,...\}$, $n = [\alpha]+1$, $f \in C^n[a,b]$ be a continuous n. order differentiable function. Right-sided and left-sided Caputo fractional derivatives are defined as follows:

$$\left({}^C D_{a+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(\tau)}{(x-\tau)^{\alpha-n+1}} d\tau, \quad x > a$$

$$\left({}^C D_{b-}^{\alpha} f\right)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{f^{(n)}(\tau)}{(\tau-x)^{\alpha-n+1}} d\tau, \quad x < b.$$

Definition 1.5. Let $\alpha > 0$, $k \geq 1$ and $\alpha \notin \{1,2,3,...\}$, $n = [\alpha]+1$, $f \in C^n[a,b]$. Caputo k-fractional derivatives over right-sided and left-sided α are defined as follows:

$$\left({}^C D_{a+}^{\alpha,k} f\right)(x) = \frac{1}{k\Gamma_k\left(n-\frac{\alpha}{k}\right)} \int_a^x \frac{f^{(n)}(\tau)}{(x-\tau)^{\frac{\alpha}{k}-n+1}} d\tau, \quad x > a \tag{1.1}$$

$$\left({}^C D_{b-}^{\alpha,k} f\right)(x) = \frac{(-1)^n}{k\Gamma_k\left(n-\frac{\alpha}{k}\right)} \int_x^b \frac{f^{(n)}(\tau)}{(\tau-x)^{\frac{\alpha}{k}-n+1}} d\tau, \quad x < b \tag{1.2}$$

where Γ_k is the representation of the gamma k function.

In the whole paper we consider $C^n[a,b]$ the space of functions $f : [a,b] \to \mathbb{R}$ which are $n-$ time differentiable and $f^{(n)}$ are continuous on $[a,b]$.

2 Main Results

In this section, we expand the Hermite-Hadamard inequality and prove this important inequality using Caputo and k-Caputo fractional derivatives.

Theorem 2.1. Let $f : [a,b] \to \mathbb{R}$ be a positive, $(n+2)$ differentiable function with $a < b$ and $f \in L_1[a,b]$. If $f^{(n+2)}$ bounded in $[a,b]$, then we have

$$\begin{aligned} &\frac{m(n-\alpha)}{2(b-a)^{n-\alpha}} \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2}-x\right)^2 \left[(x-a)^{n-\alpha-1} + (b-x)^{n-\alpha-1}\right] dx \\ \leq\ &\frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}} \left[{}^C D_{a+}^{\alpha} f(b) + (-1)^n\ {}^C D_{b-}^{\alpha} f(a)\right] - f^{(n)}\left(\frac{a+b}{2}\right) \\ \leq\ &\frac{M(n-\alpha)}{2(b-a)^{n-\alpha}} \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2}-x\right)^2 \left[(x-a)^{n-\alpha-1} + (b-x)^{n-\alpha-1}\right] dx \end{aligned} \tag{2.1}$$

and

$$
\begin{aligned}
& -\frac{M(n-\alpha)}{2(b-a)^{n-\alpha}} \int_a^{\frac{a+b}{2}} (x-a)(b-x)\left[(x-a)^{n-\alpha-1}+(b-x)^{n-\alpha-1}\right] dx \\
\leq \quad & \frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[{}^{C}D_{a+}^{\alpha} f(b)+(-1)^n \ {}^{C}D_{b-}^{\alpha} f(a)\right]-\frac{f^{(n)}(a)+f^{(n)}(b)}{2} \qquad (2.2) \\
\leq \quad & -\frac{m(n-\alpha)}{2(b-a)^{n-\alpha}} \int_a^{\frac{a+b}{2}} (x-a)(b-x)\left[(x-a)^{n-\alpha-1}+(b-x)^{n-\alpha-1}\right] dx
\end{aligned}
$$

for $\alpha>0$, where $m=\inf_{t\in[a,b]} f^{(n+2)}(t)$, $M=\sup_{t\in[a,b]} f^{(n+2)}(t)$.

Proof. We first prove the inequality (2.1), using the definition 1.4.

$$
\begin{aligned}
& \frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[{}^{C}D_{a+}^{\alpha} f(b)+(-1)^n \ {}^{C}D_{b-}^{\alpha} f(a)\right] \\
= \quad & \frac{(n-\alpha)}{2(b-a)^{n-\alpha}}\left[\int_a^b (b-x)^{n-\alpha-1} f^{(n)}(x)\, dx+\int_a^b (x-a)^{n-\alpha-1} f^{(n)}(x)\, dx\right] \\
= \quad & \frac{(n-\alpha)}{2(b-a)^{n-\alpha}} \int_a^b f^{(n)}(x)\left[(x-a)^{n-\alpha-1}+(b-x)^{n-\alpha-1}\right] dx \\
= \quad & \frac{(n-\alpha)}{2(b-a)^{n-\alpha}} \int_a^b f^{(n)}(a+b-x)\left[(x-a)^{n-\alpha-1}+(b-x)^{n-\alpha-1}\right] dx.
\end{aligned}
$$

So that we obtain

$$
\begin{aligned}
& \frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[{}^{C}D_{a+}^{\alpha} f(b)+(-1)^n \ {}^{C}D_{b-}^{\alpha} f(a)\right] \\
= \quad & \frac{(n-\alpha)}{4(b-a)^{n-\alpha}} \int_a^b \left[f^{(n)}(x)+f^{(n)}(a+b-x)\right]\left[(x-a)^{n-\alpha-1}+(b-x)^{n-\alpha-1}\right] dx
\end{aligned}
$$

which gives

$$
\begin{aligned}
& \frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[{}^{C}D_{a+}^{\alpha} f(b)+(-1)^n \ {}^{C}D_{b-}^{\alpha} f(a)\right]-f^{(n)}\left(\frac{a+b}{2}\right) \\
= \quad & \frac{(n-\alpha)}{4(b-a)^{n-\alpha}} \int_a^b \left[f^{(n)}(x)+f^{(n)}(a+b-x)-2f^{(n)}\left(\frac{a+b}{2}\right)\right]\left[(x-a)^{n-\alpha-1}+(b-x)^{n-\alpha-1}\right] dx.
\end{aligned}
$$

Since the function f is symmetric about $x=\frac{a+b}{2}$, we have

$$\frac{(n-\alpha)}{4(b-a)^{n-\alpha}}\int_a^b\left[f^{(n)}(x)+f^{(n)}(a+b-x)-2f^{(n)}\left(\frac{a+b}{2}\right)\right]\left[(x-a)^{n-\alpha-1}+(b-x)^{n-\alpha-1}\right]dx$$
$$=\frac{(n-\alpha)}{2(b-a)^{n-\alpha}}\int_a^{\frac{a+b}{2}}\left[f^{(n)}(x)+f^{(n)}(a+b-x)-2f^{(n)}\left(\frac{a+b}{2}\right)\right]\left[(x-a)^{n-\alpha-1}+(b-x)^{n-\alpha-1}\right]dx$$

which implies

$$\frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[{}^{C}D_{a^+}^{\alpha}f(b)+(-1)^n\ {}^{C}D_{b^-}^{\alpha}f(a)\right]-f^{(n)}\left(\frac{a+b}{2}\right)$$
$$=\frac{(n-\alpha)}{2(b-a)^{n-\alpha}}\int_a^{\frac{a+b}{2}}\left[f^{(n)}(x)+f^{(n)}(a+b-x)-2f^{(n)}\left(\frac{a+b}{2}\right)\right]\left[(x-a)^{n-\alpha-1}+(b-x)^{n-\alpha-1}\right]dx.$$

We can write

$$f^{(n)}(a+b-x)-f^{(n)}\left(\frac{a+b}{2}\right)=\int_{\frac{a+b}{2}}^{a+b-x}f^{(n+1)}(t)\,dt \tag{2.3}$$

and

$$f^{(n)}\left(\frac{a+b}{2}\right)-f^{(n)}(x)=\int_x^{\frac{a+b}{2}}f^{(n+1)}(t)\,dt. \tag{2.4}$$

So that we have

$$\begin{aligned}f^{(n)}(x)+f^{(n)}(a+b-x)-2f^{(n)}\left(\frac{a+b}{2}\right) &= \int_{\frac{a+b}{2}}^{a+b-x}f^{(n+1)}(t)\,dt-\int_x^{\frac{a+b}{2}}f^{(n+1)}(t)\,dt\\ &= \int_x^{\frac{a+b}{2}}f^{(n+1)}(a+b-t)\,dt-\int_x^{\frac{a+b}{2}}f^{(n+1)}(t)\,dt\\ &= \int_x^{\frac{a+b}{2}}\left[f^{(n+1)}(a+b-t)-f^{(n+1)}(t)\right]dt.\end{aligned}$$

Using (2.3) and (2.4) we have

$$f^{(n+1)}(a+b-t)-f^{(n+1)}(t)=\int_t^{a+b-t}f^{(n+2)}(y)\,dy. \tag{2.5}$$

Then for $t \in \left[a, \frac{a+b}{2}\right]$, we get

$$m(a+b-2t) \le f^{(n+1)}(a+b-t) - f^{(n+1)}(t) \le M(a+b-2t). \tag{2.6}$$

Thus we obtain

$$\int_x^{\frac{a+b}{2}} m(a+b-2t)\,dt \le f^{(n)}(x) + f^{(n)}(a+b-x) - 2f^{(n)}\left(\frac{a+b}{2}\right) \le \int_x^{\frac{a+b}{2}} M(a+b-2t)\,dt$$

and

$$m\left(\frac{a+b}{2}-x\right)^2 \le f^{(n)}(x) + f^{(n)}(a+b-x) - 2f^{(n)}\left(\frac{a+b}{2}\right) \le M\left(\frac{a+b}{2}-x\right)^2.$$

So that we have the following inequality

$$\begin{aligned}
&\frac{m(n-\alpha)}{2(b-a)^{n-\alpha}} \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2}-x\right)^2 \left[(x-a)^{n-\alpha-1} + (b-x)^{n-\alpha-1}\right] dx \\
\le\ & \frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}} \left[{}^{C}D_{a+}^{\alpha} f(b) + (-1)^n\ {}^{C}D_{b-}^{\alpha} f(a)\right] - f^{(n)}\left(\frac{a+b}{2}\right) \\
\le\ & \frac{M(n-\alpha)}{2(b-a)^{n-\alpha}} \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2}-x\right)^2 \left[(x-a)^{n-\alpha-1} + (b-x)^{n-\alpha-1}\right] dx
\end{aligned}$$

which completes the proof of (2.1). For the next part of the inequality, we have

$$\begin{aligned}
&\frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}} \left[{}^{C}D_{a+}^{\alpha} f(b) + (-1)^n\ {}^{C}D_{b-}^{\alpha} f(a)\right] - \frac{f^{(n)}(a) + f^{(n)}(b)}{2} \\
=\ & \frac{(n-\alpha)}{4(b-a)^{n-\alpha}} \int_a^b \left[f^{(n)}(x) + f^{(n)}(a+b-x) - \left(f^{(n)}(a) + f^{(n)}(b)\right)\right] \left[(x-a)^{n-\alpha-1} + (b-x)^{n-\alpha-1}\right] dx.
\end{aligned}$$

Since the function f is symmetric about $x = \frac{a+b}{2}$, we get

$$\begin{aligned}
&\frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}} \left[{}^{C}D_{a+}^{\alpha} f(b) + (-1)^n\ {}^{C}D_{b-}^{\alpha} f(a)\right] - \frac{f^{(n)}(a) + f^{(n)}(b)}{2} \\
=\ & \frac{(n-\alpha)}{2(b-a)^{n-\alpha}} \int_a^{\frac{a+b}{2}} \left[f^{(n)}(x) + f^{(n)}(a+b-x) - \left(f^{(n)}(a) + f^{(n)}(b)\right)\right] \left[(x-a)^{n-\alpha-1} + (b-x)^{n-\alpha-1}\right] dx.
\end{aligned}$$

As we find in (2.3) and (2.4), we can write

$$f^{(n)}(b) - f^{(n)}(a+b-x) = \int_{a+b-x}^{b} f^{(n+1)}(t)\,dt \tag{2.7}$$

and

$$f^{(n)}(x) - f^{(n)}(a) = \int_a^x f^{(n+1)}(t)\,dt. \tag{2.8}$$

Then using (2.7) and (2.8), we have

$$\begin{aligned} f^{(n)}(x) + f^{(n)}(a+b-x) - \left(f^{(n)}(a) + f^{(n)}(b)\right) &= \int_a^x f^{(n+1)}(t)\,dt - \int_{a+b-x}^b f^{(n+1)}(t)\,dt \\ &= \int_a^x f^{(n+1)}(t)\,dt - \int_a^x f^{(n+1)}(a+b-t)\,dt \\ &= -\int_a^x \left[f^{(n+1)}(a+b-t) - f^{(n+1)}(t)\right] dt. \end{aligned}$$

We also have from (2.5) and (2.6)

$$-\int_a^x M(a+b-2t)\,dt \le f^{(n)}(x) + f^{(n)}(a+b-x) - \left(f^{(n)}(a) + f^{(n)}(b)\right) \le -\int_a^x m(a+b-2t)\,dt.$$

If we integrate, we get

$$-M(x-a)(b-x) \le f^{(n)}(x) + f^{(n)}(a+b-x) - \left(f^{(n)}(a) + f^{(n)}(b)\right) \le -m(x-a)(b-x)$$

and

$$\begin{aligned} &-\frac{M(n-\alpha)}{2(b-a)^{n-\alpha}} \int_a^{\frac{a+b}{2}} (x-a)(b-x)\left[(x-a)^{n-\alpha-1} + (b-x)^{n-\alpha-1}\right] dx \\ \le\ & \frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}} \left[{}^{C}D_{a^+}^{\alpha} f(b) + (-1)^n\ {}^{C}D_{b^-}^{\alpha} f(a)\right] - \frac{f^{(n)}(a) + f^{(n)}(b)}{2} \\ \le\ & -\frac{m(n-\alpha)}{2(b-a)^{n-\alpha}} \int_a^{\frac{a+b}{2}} (x-a)(b-x)\left[(x-a)^{n-\alpha-1} + (b-x)^{n-\alpha-1}\right] dx. \end{aligned}$$

So we have completed the proof of (2.2). Q.E.D.

Theorem 2.2. Let $f : [a,b] \to \mathbb{R}$ be a positive, $(n+2)$ differentiable function with $a < b$ and $f \in L_1[a,b]$. If $f^{(n+1)}(a+b-x) \ge f^{(n+1)}(x)$ for all $x \in \left[a, \frac{a+b}{2}\right]$ .Then the following fractional inequalities hold

$$f^{(n)}\left(\frac{a+b}{2}\right) \le \frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}} \left[{}^{C}D_{a^+}^{\alpha} f(b) + (-1)^n\ {}^{C}D_{b^-}^{\alpha} f(a)\right] \le \frac{f^{(n)}(a) + f^{(n)}(b)}{2}$$

for $\alpha > 0$.

Proof. We have

$$\begin{aligned}
&\frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[{}^{C}D_{a+}^{\alpha}f(b)+(-1)^{n}\ {}^{C}D_{b-}^{\alpha}f(a)\right]-f^{(n)}\left(\frac{a+b}{2}\right)\\
=\ &\frac{(n-\alpha)}{2(b-a)^{n-\alpha}}\int_{a}^{\frac{a+b}{2}}\left[f^{(n)}(x)+f^{(n)}(a+b-x)-2f^{(n)}\left(\frac{a+b}{2}\right)\right]\left[(x-a)^{n-\alpha-1}+(b-x)^{n-\alpha-1}\right]dx\\
=\ &\frac{(n-\alpha)}{2(b-a)^{n-\alpha}}\int_{a}^{\frac{a+b}{2}}\left[\int_{x}^{\frac{a+b}{2}}\left[f^{(n+1)}(a+b-t)-f^{(n+1)}(t)\right]dt\right]\left[(x-a)^{n-\alpha-1}+(b-x)^{n-\alpha-1}\right]dx\\
\geq\ &0.
\end{aligned}$$

Then as in the proof of the Theorem 2.1, we can write

$$\begin{aligned}
&\frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[{}^{C}D_{\alpha+}^{\alpha}f(b)+(-1)^{n}\ {}^{C}D_{b-}^{\alpha}f(a)\right]-\frac{f^{(n)}(a)+f^{(n)}(b)}{2}\\
=\ &\frac{(n-\alpha)}{2(b-a)^{n-\alpha}}\int_{a}^{\frac{a+b}{2}}\left[-\int_{a}^{x}\left[f^{(n+1)}(a+b-t)-f^{(n+1)}(t)\right]dt\right]\left[(x-a)^{n-\alpha-1}+(b-x)^{n-\alpha-1}\right]dx\\
\leq\ &0.
\end{aligned}$$

This completes the proof. Q.E.D.

Theorem 2.3. Let $f:[a,b]\to\mathbb{R}$ be a positive, $(n+2)$ differentiable function with $a<b$ and $f\in L_1[a,b]$. If $f^{(n+2)}$ is bounded in $[a,b]$, then we have

$$\begin{aligned}
&\frac{m\left(n-\frac{\alpha}{k}\right)}{2(b-a)^{n-\frac{\alpha}{k}}}\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^{2}\left[(x-a)^{n-\frac{\alpha}{k}-1}+(b-x)^{n-\frac{\alpha}{k}-1}\right]dx\\
\leq\ &\frac{\Gamma\left(n-\frac{\alpha}{k}+1\right)}{2(b-a)^{n-\frac{\alpha}{k}}}\left[{}^{C}D_{a^{+}}^{\alpha,k}f(b)+(-1)^{n}\ {}^{C}D_{b^{-}}^{\alpha,k}f(a)\right]-f^{(n)}\left(\frac{a+b}{2}\right) \qquad (2.9)\\
\leq\ &\frac{M\left(n-\frac{\alpha}{k}\right)}{2(b-a)^{n-\frac{\alpha}{k}}}\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^{2}\left[(x-a)^{n-\frac{\alpha}{k}-1}+(b-x)^{n-\frac{\alpha}{k}-1}\right]dx
\end{aligned}$$

and

$$
\begin{aligned}
&-\frac{M\left(n-\frac{\alpha}{k}\right)}{2\left(b-a\right)^{n-\frac{\alpha}{k}}}\int\limits_{a}^{\frac{a+b}{2}}\left(x-a\right)\left(b-x\right)\left[\left(x-a\right)^{n-\frac{\alpha}{k}-1}+\left(b-x\right)^{n-\frac{\alpha}{k}-1}\right]dx\\
\leq\;&\frac{\Gamma\left(n-\frac{\alpha}{k}+1\right)}{2\left(b-a\right)^{n-\frac{\alpha}{k}}}\left[{}^{C}D_{a^{+}}^{\alpha,k}f\left(b\right)+\left(-1\right)^{n}\ {}^{C}D_{b^{-}}^{\alpha,k}f\left(a\right)\right]-\frac{f^{(n)}\left(a\right)+f^{(n)}\left(b\right)}{2}\\
\leq\;&-\frac{m\left(n-\frac{\alpha}{k}\right)}{2\left(b-a\right)^{n-\frac{\alpha}{k}}}\int\limits_{a}^{\frac{a+b}{2}}\left(x-a\right)\left(b-x\right)\left[\left(x-a\right)^{n-\frac{\alpha}{k}-1}+\left(b-x\right)^{n-\frac{\alpha}{k}-1}\right]dx
\end{aligned}
$$

for $\alpha>0$, where $m=\inf_{t\in[a,b]}f^{(n+2)}\left(t\right)$, $M=\sup_{t\in[a,b]}f^{(n+2)}\left(t\right)$.

Proof. As in Theorem 2.1, we first prove (2.9). Using the definition 1.6, we have

$$
\begin{aligned}
&\frac{\Gamma\left(n-\frac{\alpha}{k}+1\right)}{2\left(b-a\right)^{n-\frac{\alpha}{k}}}\left[{}^{C}D_{a^{+}}^{\alpha,k}f\left(b\right)+\left(-1\right)^{n}\ {}^{C}D_{b^{-}}^{\alpha,k}f\left(a\right)\right]\\
=\;&\frac{\left(n-\frac{\alpha}{k}\right)}{2\left(b-a\right)^{n-\frac{\alpha}{k}}}\left[\int\limits_{a}^{b}\left(b-x\right)^{n-\frac{\alpha}{k}-1}f^{(n)}\left(x\right)dx+\int\limits_{a}^{b}\left(x-a\right)^{n-\frac{\alpha}{k}-1}f^{(n)}\left(x\right)dx\right]\\
=\;&\frac{\left(n-\frac{\alpha}{k}\right)}{2\left(b-a\right)^{n-\frac{\alpha}{k}}}\left[\int\limits_{a}^{b}f^{(n)}\left(x\right)\left[\left(x-a\right)^{n-\frac{\alpha}{k}-1}+\left(b-x\right)^{n-\frac{\alpha}{k}-1}\right]dx\right]\\
=\;&\frac{\left(n-\frac{\alpha}{k}\right)}{2\left(b-a\right)^{n-\frac{\alpha}{k}}}\int\limits_{a}^{b}f^{(n)}\left(a+b-x\right)\left[\left(x-a\right)^{n-\frac{\alpha}{k}-1}+\left(b-x\right)^{n-\frac{\alpha}{k}-1}\right]dx.
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
&\frac{\Gamma\left(n-\frac{\alpha}{k}+1\right)}{2\left(b-a\right)^{n-\frac{\alpha}{k}}}\left[{}^{C}D_{a^{+}}^{\alpha,k}f\left(b\right)+\left(-1\right)^{n}\ {}^{C}D_{b^{-}}^{\alpha,k}f\left(a\right)\right]-f^{(n)}\left(\frac{a+b}{2}\right)\\
=\;&\frac{\left(n-\frac{\alpha}{k}\right)}{4\left(b-a\right)^{n-\frac{\alpha}{k}}}\int\limits_{a}^{b}\left[f^{(n)}\left(x\right)+f^{(n)}\left(a+b-x\right)\right]\left[\left(x-a\right)^{n-\frac{\alpha}{k}-1}+\left(b-x\right)^{n-\frac{\alpha}{k}-1}\right]dx
\end{aligned}
$$

which gives

$$
\begin{aligned}
&\frac{\Gamma\left(n-\frac{\alpha}{k}+1\right)}{2\left(b-a\right)^{n-\frac{\alpha}{k}}}\left[{}^{C}D_{a^{+}}^{\alpha,k}f\left(b\right)+\left(-1\right)^{n}\ {}^{C}D_{b^{-}}^{\alpha,k}f\left(a\right)\right]-f^{(n)}\left(\frac{a+b}{2}\right)\\
=\;&\frac{\left(n-\frac{\alpha}{k}\right)}{4\left(b-a\right)^{n-\frac{\alpha}{k}}}\int\limits_{a}^{b}\left[f^{(n)}\left(x\right)+f^{(n)}\left(a+b-x\right)-2f^{(n)}\left(\frac{a+b}{2}\right)\right]\left[\left(x-a\right)^{n-\frac{\alpha}{k}-1}+\left(b-x\right)^{n-\frac{\alpha}{k}-1}\right]dx.
\end{aligned}
$$

The function f is symmetric about $x=\frac{a+b}{2}$, we have

$$\frac{\left(n-\frac{\alpha}{k}\right)}{4\left(b-a\right)^{n-\frac{\alpha}{k}}}\int\limits_{a}^{b}\left[f^{(n)}\left(x\right)+f^{(n)}\left(a+b-x\right)-2f^{(n)}\left(\frac{a+b}{2}\right)\right]\left[\left(x-a\right)^{n-\frac{\alpha}{k}-1}+\left(b-x\right)^{n-\frac{\alpha}{k}-1}\right]dx$$
$$=\frac{\left(n-\frac{\alpha}{k}\right)}{2\left(b-a\right)^{n-\frac{\alpha}{k}}}\int\limits_{a}^{\frac{a+b}{2}}\left[f^{(n)}\left(x\right)+f^{(n)}\left(a+b-x\right)-2f^{(n)}\left(\frac{a+b}{2}\right)\right]\left[\left(x-a\right)^{n-\frac{\alpha}{k}-1}+\left(b-x\right)^{n-\frac{\alpha}{k}-1}\right]dx$$

which implies

$$\frac{\Gamma\left(n-\frac{\alpha}{k}+1\right)}{2\left(b-a\right)^{n-\frac{\alpha}{k}}}\left[{}^{C}D_{a^{+}}^{\alpha,k}f\left(b\right)+\left(-1\right)^{n}\ {}^{C}D_{b^{-}}^{\alpha,k}f\left(a\right)\right]-f^{(n)}\left(\frac{a+b}{2}\right)$$
$$=\frac{\left(n-\frac{\alpha}{k}\right)}{2\left(b-a\right)^{n-\frac{\alpha}{k}}}\int\limits_{a}^{\frac{a+b}{2}}\left[f^{(n)}\left(x\right)+f^{(n)}\left(a+b-x\right)-2f^{(n)}\left(\frac{a+b}{2}\right)\right]\left[\left(x-a\right)^{n-\frac{\alpha}{k}-1}+\left(b-x\right)^{n-\frac{\alpha}{k}-1}\right]dx.$$

Using (2.3), (2.4), (2.5) and (2.6), we obtain

$$\frac{m\left(n-\frac{\alpha}{k}\right)}{2\left(b-a\right)^{n-\frac{\alpha}{k}}}\int\limits_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^{2}\left[\left(x-a\right)^{n-\frac{\alpha}{k}-1}+\left(b-x\right)^{n-\frac{\alpha}{k}-1}\right]dx$$
$$\leq\frac{\Gamma\left(n-\frac{\alpha}{k}+1\right)}{2\left(b-a\right)^{n-\frac{\alpha}{k}}}\left[{}^{C}D_{\alpha^{+}}^{\alpha,k}f\left(b\right)+\left(-1\right)^{n}\ {}^{C}D_{b^{-}}^{\alpha,k}f\left(a\right)\right]-f^{(n)}\left(\frac{a+b}{2}\right)$$
$$\leq\frac{M\left(n-\frac{\alpha}{k}\right)}{2\left(b-a\right)^{n-\frac{\alpha}{k}}}\int\limits_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^{2}\left[\left(x-a\right)^{n-\frac{\alpha}{k}-1}+\left(b-x\right)^{n-\frac{\alpha}{k}-1}\right]dx$$

which completes the proof of (2.9). For the second part of the inequality, we have

$$\frac{\Gamma\left(n-\frac{\alpha}{k}+1\right)}{2\left(b-a\right)^{n-\frac{\alpha}{k}}}\left[{}^{C}D_{a^{+}}^{\alpha,k}f\left(b\right)+\left(-1\right)^{n}\ {}^{C}D_{b^{-}}^{\alpha,k}f\left(a\right)\right]-\frac{f^{(n)}\left(a\right)+f^{(n)}\left(b\right)}{2}$$
$$=\frac{\left(n-\frac{\alpha}{k}\right)}{4\left(b-a\right)^{n-\frac{\alpha}{k}}}\int\limits_{a}^{b}\left[f^{(n)}\left(x\right)+f^{(n)}\left(a+b-x\right)-\left(f^{(n)}\left(a\right)+f^{(n)}\left(b\right)\right)\right]\left[\left(x-a\right)^{n-\frac{\alpha}{k}-1}+\left(b-x\right)^{n-\frac{\alpha}{k}-1}\right]dx.$$

Since the function f is symmetric about $x=\frac{a+b}{2}$, we get

$$\frac{\Gamma\left(n-\frac{\alpha}{k}+1\right)}{2\left(b-a\right)^{n-\frac{\alpha}{k}}}\left[{}^{C}D_{a^{+}}^{\alpha,k}f\left(b\right)+\left(-1\right)^{n}\ {}^{C}D_{b^{-}}^{\alpha,k}f\left(a\right)\right]-\frac{f^{(n)}\left(a\right)+f^{(n)}\left(b\right)}{2}$$
$$=\frac{\left(n-\frac{\alpha}{k}\right)}{2\left(b-a\right)^{n-\frac{\alpha}{k}}}\int\limits_{a}^{\frac{a+b}{2}}\left[f^{(n)}\left(x\right)+f^{(n)}\left(a+b-x\right)-\left(f^{(n)}\left(a\right)+f^{(n)}\left(b\right)\right)\right]\left[\left(x-a\right)^{n-\frac{\alpha}{k}-1}+\left(b-x\right)^{n-\frac{\alpha}{k}-1}\right]dx.$$

As we find in the proof of the Theorem 2.1, we have from (2.3), (2.4),(2.5) and (2.6)

$$-\int_a^x M(a+b-2t)\,dt \le f^{(n)}(x)+f^{(n)}(a+b-x)-\left(f^{(n)}(a)+f^{(n)}(b)\right) \le -\int_a^x m(a+b-2t)\,dt.$$

$$-M(x-a)(b-x) \le f^{(n)}(x)+f^{(n)}(a+b-x)-\left(f^{(n)}(a)+f^{(n)}(b)\right) \le -m(x-a)(b-x)$$

and

$$\begin{aligned} & -\frac{M\left(n-\frac{\alpha}{k}\right)}{2(b-a)^{n-\frac{\alpha}{k}}}\int_a^{\frac{a+b}{2}}(x-a)(b-x)\left[(x-a)^{n-\frac{\alpha}{k}-1}+(b-x)^{n-\frac{\alpha}{k}-1}\right]dx \\ \le\; & \frac{\Gamma\left(n-\frac{\alpha}{k}+1\right)}{2(b-a)^{n-\frac{\alpha}{k}}}\left[{}^{C}D_{a^+}^{\alpha,k}f(b)+(-1)^n\;{}^{C}D_{b^-}^{\alpha,k}f(a)\right]-\frac{f^{(n)}(a)+f^{(n)}(b)}{2} \\ \le\; & -\frac{m\left(n-\frac{\alpha}{k}\right)}{2(b-a)^{n-\frac{\alpha}{k}}}\int_a^{\frac{a+b}{2}}(x-a)(b-x)\left[(x-a)^{n-\frac{\alpha}{k}-1}+(b-x)^{n-\frac{\alpha}{k}-1}\right]dx. \end{aligned}$$

So that we have completed the proof. Q.E.D.

Theorem 2.4. Let $f:[a,b]\to\mathbb{R}$ be a positive, differentiable function with $a<b$ and $f\in L_1[a,b]$. If the inequality $f^{(n+1)}(a+b-x)\ge f^{(n+1)}(x)$ holds for all $x\in\left[a,\frac{a+b}{2}\right]$, then the following fractional inequalities hold

$$f^{(n)}\left(\frac{a+b}{2}\right)\le\frac{\Gamma\left(n-\frac{\alpha}{k}+1\right)}{2(b-a)^{n-\frac{\alpha}{k}}}\left[{}^{C}D_{a^+}^{\alpha,k}f(b)+(-1)^n\;{}^{C}D_{b^-}^{\alpha,k}f(a)\right]\le\frac{f^{(n)}(a)+f^{(n)}(b)}{2}$$

for $\alpha>0$.

Proof. Using the definition of k-Caputo derivative in (1.1) and (1.2) we have

$$\begin{aligned} & \frac{\Gamma\left(n-\frac{\alpha}{k}+1\right)}{2(b-a)^{n-\frac{\alpha}{k}}}\left[{}^{C}D_{a^+}^{\alpha,k}f(b)+(-1)^n\;{}^{C}D_{b^-}^{\alpha,k}f(a)\right]-f^{(n)}\left(\frac{a+b}{2}\right) \\ =\; & \frac{\left(n-\frac{\alpha}{k}\right)}{2(b-a)^{n-\frac{\alpha}{k}}}\int_a^{\frac{a+b}{2}}\left[f^{(n)}(x)+f^{(n)}(a+b-x)-2f^{(n)}\left(\frac{a+b}{2}\right)\right]\left[(x-a)^{n-\frac{\alpha}{k}-1}+(b-x)^{n-\frac{\alpha}{k}-1}\right]dx \\ =\; & \frac{\left(n-\frac{\alpha}{k}\right)}{2(b-a)^{n-\frac{\alpha}{k}}}\int_a^{\frac{a+b}{2}}\left[\int_x^{\frac{a+b}{2}}\left[f^{(n+1)}(a+b-t)-f^{(n+1)}(t)\,dt\right]\right]\left[(x-a)^{n-\frac{\alpha}{k}-1}+(b-x)^{n-\frac{\alpha}{k}-1}\right]dx \\ \ge\; & 0. \end{aligned}$$

$$\begin{aligned} & \frac{\Gamma\left(n-\frac{\alpha}{k}+1\right)}{2\left(b-a\right)^{n-\frac{\alpha}{k}}}\left[{}^{C}D_{a+}^{\alpha,k}f\left(b\right)+\left(-1\right)^{n}\ {}^{C}D_{b-}^{\alpha,k}f\left(a\right)\right]-\frac{f^{(n)}\left(a\right)+f^{(n)}\left(b\right)}{2} \\ = & \frac{\left(n-\frac{\alpha}{k}\right)}{2\left(b-a\right)^{n-\frac{\alpha}{k}}}\int_{a}^{\frac{a+b}{2}}\left[-\int_{a}^{x}\left[f^{(n+1)}\left(a+b-t\right)-f^{(n+1)}\left(t\right)dt\right]\right]\left[\left(x-a\right)^{n-\frac{\alpha}{k}-1}+\left(b-x\right)^{n-\frac{\alpha}{k}-1}\right]dx \\ \leq & \ 0. \end{aligned}$$

Q.E.D.

Theorem 2.5. Let $f:[a,b]\rightarrow\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f^{(n+1)}\in L[a,b]$, then the following fractional equality holds:

$$\frac{f^{(n)}\left(a\right)+f^{(n)}\left(b\right)}{2}-\frac{\Gamma\left(n-\alpha+1\right)}{2\left(b-a\right)^{n-\alpha}}\left[{}^{C}D_{a+}^{\alpha}f\left(b\right)+\left(-1\right)^{n}\ {}^{C}D_{b-}^{\alpha}f\left(a\right)\right]$$
$$=\frac{b-a}{2}\int_{0}^{1}\left[\left(1-t\right)^{n-\alpha}-t^{n-\alpha}\right]\ f^{(n+1)}\left(ta+\left(1-t\right)b\right)dt.$$

Proof. It suffices to note that

$$\begin{aligned} I &= \int_{0}^{1}\left[\left(1-t\right)^{n-\alpha}-t^{n-\alpha}\right]f^{(n+1)}\left(ta+\left(1-t\right)b\right)dt \\ &= \int_{0}^{1}\left(1-t\right)^{n-\alpha}f^{(n+1)}\left(ta+\left(1-t\right)b\right)dt-\int_{0}^{1}t^{n-\alpha}f^{(n+1)}\left(ta+\left(1-t\right)b\right)dt=I_{1}+I_{2}. \end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned} I_{1} &= \int_{0}^{1}\left(1-t\right)^{n-\alpha}f^{(n+1)}\left(ta+\left(1-t\right)b\right)dt \\ &= \left(1-t\right)^{n-\alpha}\frac{f^{(n)}\left(ta+\left(1-t\right)b\right)}{a-b}\Bigg|_{0}^{1}+\int_{0}^{1}\left(n-\alpha\right)\left(1-t\right)^{n-\alpha-1}\frac{f^{(n)}\left(ta+\left(1-t\right)b\right)}{a-b}dt \\ &= \frac{f^{(n)}\left(b\right)}{b-a}-\frac{\left(n-\alpha\right)}{b-a}\int_{b}^{a}\left(\frac{a-x}{a-b}\right)^{n-\alpha-1}\frac{f^{(n)}\left(x\right)}{a-b}dx \\ &= \frac{f^{(n)}\left(b\right)}{b-a}-\frac{\Gamma\left(n-\alpha+1\right)}{\left(b-a\right)^{n-\alpha+1}}\ {}^{C}D_{b-}^{\alpha}f\left(a\right) \end{aligned}$$

and similarly we get,

$$\begin{aligned} I_2 &= -\int_0^1 t^{n-\alpha} f^{(n+1)}(ta+(1-t)b)\,dt \\ &= -\frac{t^{n-\alpha} f^{(n)}(ta+(1-t)b)}{a-b}\Bigg|_0^1 + (n-\alpha)\int_0^1 t^{n-\alpha-1}\frac{f^{(n)}(ta+(1-t)b)}{a-b}dt \\ &= \frac{f^{(n)}(a)}{b-a} - \frac{(n-\alpha)}{b-a}\int_b^a \left(\frac{b-x}{b-a}\right)^{n-\alpha-1}\frac{f^{(n)}(x)}{a-b}dx \\ &= \frac{f^{(n)}(a)}{b-a} - \frac{\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha+1}}\ {}^{C}D^{\alpha}_{a^+} f(b). \end{aligned}$$

As a result we can write

$$I = \frac{f^{(n)}(a)+f^{(n)}(b)}{b-a} - \frac{\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha+1}}\left[{}^{C}D^{\alpha}_{a^+} f(b) + (-1)^n\ {}^{C}D^{\alpha}_{b^-} f(a)\right].$$

Thus multiplying both sides by $\frac{b-a}{2}$, we have

$$\begin{aligned} &\frac{f^{(n)}(a)+f^{(n)}(b)}{2} - \frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[{}^{C}D^{\alpha}_{a^+} f(b) + (-1)^n\ {}^{C}D^{\alpha}_{b^-} f(a)\right] \\ &\qquad = \frac{b-a}{2}\int_0^1 \left[(1-t)^{n-\alpha} - t^{n-\alpha}\right] f^{(n+1)}(ta+(1-t)b)\,dt. \end{aligned}$$

Q.E.D.

Theorem 2.6. Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $\left|f^{(n+1)}\right|$ is convex on $[a,b]$, then the following fractional inequality holds:

$$\begin{aligned} &\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2} - \frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[{}^{C}D^{\alpha}_{a^+} f(b) + (-1)^n\ {}^{C}D^{\alpha}_{b^-} f(a)\right]\right| \\ &\qquad \le \frac{b-a}{2(n-\alpha+1)}\left(1-\frac{1}{2^{n-\alpha}}\right)\left[f^{(n+1)}(a) + f^{(n+1)}(b)\right] \end{aligned}$$

Proof. Using Theorem 2.5 and the convexity of $\left|f^{(n+1)}\right|$, we find

$$
\begin{aligned}
&\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}-\frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[{}^{C}D_{a+}^{\alpha}f(b)+(-1)^{n}\ {}^{C}D_{b-}^{\alpha}f(a)\right]\right| \\
\leq\ & \frac{b-a}{2}\int_{0}^{1}\left|(1-t)^{n-\alpha}-t^{n-\alpha}\right|\ \left|f^{(n+1)}(ta+(1-t)b)\right|\ dt \\
\leq\ & \frac{b-a}{2}\int_{0}^{1}\left|(1-t)^{n-\alpha}-t^{n-\alpha}\right|\left[\ t\ \left|f^{(n+1)}(a)\right|+(1-t)\left|f^{(n+1)}(b)\right|\right]dt \\
=\ & \frac{b-a}{2}\left\{\int_{0}^{\frac{1}{2}}\ \left|(1-t)^{n-\alpha}-t^{n-\alpha}\right|\ \left[t\ \left|f^{(n+1)}(a)\right|+(1-t)\ \left|f^{(n+1)}(b)\right|\right]dt\right\} \\
+\ & \frac{b-a}{2}\left\{\int_{\frac{1}{2}}^{1}\left|t^{n-\alpha}-(1-t)^{n-\alpha}\right|\ \left[t\ \left|f^{(n+1)}(a)\right|+(1-t)\ \left|f^{(n+1)}(b)\right|\right]dt\right\} \\
=\ & \frac{b-a}{2}\left(K_1+K_2\right).
\end{aligned}
$$

Calculating K_1 and K_2, we have

$$
\begin{aligned}
K_1 &= \left|f^{(n+1)}(a)\right|\left[\int_{0}^{\frac{1}{2}}t(1-t)^{n-\alpha}dt-\int_{0}^{1}t^{n-\alpha+1}dt\right]+\left|f^{(n+1)}(b)\right|\left[\int_{0}^{\frac{1}{2}}(1-t)^{n-\alpha+1}dt-\int_{0}^{\frac{1}{2}}(1-t)t^{n-\alpha}dt\right] \\
&= \left|f^{(n+1)}(a)\right|\left[\frac{1}{(n-\alpha+1)(n-\alpha+2)}-\frac{\left(\frac{1}{2}\right)^{n-\alpha+1}}{n-\alpha+1}\right]+\left|f^{(n+1)}(b)\right|\left[\frac{1}{n-\alpha+2}-\frac{\left(\frac{1}{2}\right)^{n-\alpha+1}}{n-\alpha+1}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
K_2 &= \left|f^{(n+1)}(a)\right|\left[\int_{\frac{1}{2}}^{1}t^{n-\alpha+1}dt-\int_{\frac{1}{2}}^{1}t(1-t)^{n-\alpha}dt\right]+\left|f^{(n+1)}(b)\right|\left[\int_{\frac{1}{2}}^{1}(1-t)t^{n-\alpha}dt-\int_{\frac{1}{2}}^{1}(1-t)^{n-\alpha+1}dt\right] \\
&= \left|f^{(n+1)}(a)\right|\left[\frac{1}{n-\alpha+2}-\frac{\left(\frac{1}{2}\right)^{n-\alpha+1}}{n-\alpha+1}\right]+\left|f^{(n+1)}(b)\right|\left[\frac{1}{(n-\alpha+1)(n-\alpha+2)}-\frac{\left(\frac{1}{2}\right)^{n-\alpha+1}}{n-\alpha+1}\right].
\end{aligned}
$$

Thus substituting K_1 and K_2, we obtain the inequality

$$
\begin{aligned}
&\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}-\frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[{}^{C}D_{a+}^{\alpha}f(b)+(-1)^{n}\ {}^{C}D_{b-}^{\alpha}f(a)\right]\right| \\
&\qquad\leq\frac{b-a}{2(n-\alpha+1)}\left(1-\frac{1}{2^{n-\alpha}}\right)\left[f^{(n+1)}(a)+f^{(n+1)}(b)\right].
\end{aligned}
$$

Q.E.D.

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